Section 5.4
Generating Functions

Generating functions are useful for manipulating sequences and therefore for solving counting problems.

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Definition: Let S = {a_0, a_1, a_2, a_3, ...} be an (infinite) sequence of real numbers. Then the *generating function G(x),* of S is the series

$$G(x) = \sum_{k=0}^{\infty} a_k x^k$$

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Note: For students of calculus -

> There is no issue of convergence here. The variable x and its powers allow us to assign a position to the numbers a_k. We will use the closed form expressions which represent G if the series converges but for symbol manipulation purposes only.

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Examples:

• Let S = {1, 1, 1, 1, ... }. Then

$$G(x) = 1 + x + x^2 + x^3 + \ldots + x^k + \ldots = \sum_{k=0}^{\infty} x^k$$

- Let S = {0, 1, 2, 3, 4, 5, ...}. Then

$$G(x) = \sum_{k=0}^{\infty} kx^k$$

- If S is the finite sequence {1,1,1} then we pad S with an infinite number of zeros {1, 1, 1, 0, 0, ...} and we have

$$G(x) = 1 + x + x^2 = \frac{(1-x^3)}{(1-x)},$$

a closed form for G.

The Binomial Theorem Revisited

For producing expressions for G(x), recall the *binomial theorem*:

$$(a+b)^n = \frac{a^n}{0!} + \frac{na^{n-1}b}{1!} + \frac{n(n-1)a^{n-2}b^2}{2!} + \ldots$$

$$+ \frac{n(n-1)\ldots(n-k+1)a^{n-k}b^k}{k!} + \ldots + \frac{n(n-1)\ldots(n-n+`1)a^{n-n}b^n}{n!}$$

which terminates when n is an integer to produce a finite sum.

However, the formula can also be extended to include the case when n is <u>not</u> an integer:

  • the sum does not terminate

  • useful for producing expressions for generating functions.

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Example:

We apply the procedure to the expression

$$\frac{1}{(1-3x^2)}$$

where we let $a = 1, b = (-3x^2)$, and $n = -1$:

$$\frac{1}{(1-3x^2)} = (1-3x^2)^{-1} = \frac{1^{-1}}{0!} + \frac{(-1)1^{-2}(-3x^2)}{1!} +$$

$$\frac{(-1)(-2)1^{-3}(-3x^2)^2}{2!} + \frac{(-1)(-2)(-3)1^{-4}(-3x^2)^3}{3!} + ... + (3x^2)^k + ...$$

Since there are no odd powers of x, the expression is the generating function for the sequence

$$S = \{1, 0, 3, 0, 3^2, 0, 3^3, 0, ...\}$$

Manipulation of Generating Functions

Let

$$F(x) = \sum_{k=0}^{\infty} a_k x^k \text{ and } G(x) = \sum_{k=0}^{\infty} b_k x^k.$$

Then

• Sum

$$F(x) + G(x) = \sum_{k=0}^{\infty} (a_k + b_k)x^k$$

• Product

$$F(x)G(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j} \right)x^k$$

To derive the expression we multiply each term of F by all terms of G and place the coefficients of like powers of x in the same column:

$$(a_0 + a_1x + a_2x^2 + a_3x^3 + ...)(b_0 + b_1x + b_2x^2 + ...)$$

x^0	x^1	x^2	$x^3 ...$
a_0b_0	a_0b_1	a_0b_2	a_0b_3
	a_1b_0	a_1b_1	a_1b_2
		a_2b_0	a_2b_1
			a_3b_0

$$\sum_{j=0}^{0} a_jb_{0-j} \qquad \sum_{j=0}^{2} a_jb_{1-j} \qquad \sum_{j=0}^{2} a_jb_{2-j} \qquad \sum_{j=0}^{3} a_jb_{3-j} \ ...$$

Example:

A expression for the generating function of the sequence S = {1, 1, 1, 1, 1, 1, ...} is

$$\frac{1}{(1-x)}$$

Hence, the expression

$$\frac{1}{(1-x)^2}$$

is the generating function for the sequence

$$S = \{1, 2, 3, 4, ..., k+1, ...\}.$$

Counting with Generating Functions

Recall that

$$x^a x^b = x^{a+b}$$

and therefore if

$$S = \{a_0, a_1, a_2, a_3, \ldots\} \text{ and } T = \{b_0, b_1, b_2, b_3, \ldots\}$$

there will be a contribution of $a_m b_n$ to the m + n power of x in the product of the generating functions for S and T.

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Example:

Suppose we wish to count the integer solutions to a + b = 10 but a and b are constrained by

$$1 \quad a \quad 6 \text{ and } 3 \quad b \quad 9$$

There are 6 possible solutions:

$$1 + 9$$
$$2 + 8$$
$$3 + 7$$
$$4 + 6$$
$$5 + 5$$
$$6 + 4$$

where the first number is a and the second is b.

If we construct the sequences

$$A = \{0, 1, 1, 1, 1, 1, 1, 0, 0, ...\}$$

where $a_k = 1$ if k is a possible value of a and

$$B = \{0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 0, ...\}$$

where $b_k = 1$ if k is a possible value of b then the product of the two generating functions for A and B will have the total possible solutions to

$$a + b = 10$$

as a coefficient of x^{10}

$$\sum_{k=0}^{10} a_k b_{10-k}$$

$$= 0 + 1 + 1 + 1 + 1 + 1 + 1 + 0 + 0 + 0 + 0 = 6$$

Solving Recurrence Relations

First some observations:

If

$$G(x) = \sum_{k=0}^{\infty} a_k x^k$$

then

$$xG(x) = \sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{k=1}^{\infty} a_{k-1} x^{k}$$

and

$$x^2 G(x) = \sum_{k=0}^{\infty} a_k x^{k+2} = \sum_{k=2}^{\infty} a_{k-2} x^{k}$$

etc.

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Example:

Solve the nonhomogeneous recurrence system

$$a_n = 3a_{n-2} + 1, a_0 = a_1 = 1.$$

Solution:

Multiply each term on both sides of the equation by x^n and sum from 2 to infinity to produce:

$$\sum_{n=2}^{\infty} a_n x^{n} = 3 \sum_{n=2}^{\infty} a_{n-2} x^{n} + \sum_{n=2}^{\infty} x^{n}$$

Now adding $a_0 + a_1 x$ to both sides and using the initial conditions we have

$$a_0 + a_1 x + \sum_{n=2} a_n x^n = 3 \sum_{n=2} a_{n-2} x^n + \sum_{n=2} x^n + 1 + x$$

which gives

$$\sum_{n=0} a_n x^n = 3 \sum_{n=2} a_{n-2} x^n + \sum_{n=0} x^n$$

or

$$G(x) - 3x^2 G(x) = \frac{1}{(1-x)}$$

or

$$G(x) = \frac{1}{(1-3x^2)} \frac{1}{(1-x)}$$

We leave it to the student to find the terms of the sequence represented by G.

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