Discrete Mathematics 2001/02

MA 311

Exercises 4

1 Suppose the sequence $(p_n)_{n=0}^{\infty}$ has generating function $f(x)$, and the sequence $(q_n)_{n=0}^{\infty}$ has generating function $g(x)$. Prove the following assertions:

(a) The function $f(x)+g(x)$ generates the sequence $(a_n)_{n=0}^{\infty}$ with $a_n = p_n + q_n$ for $n \geq 0$.

(b) For $\alpha \in \mathbb{R}$, the function $\alpha\, f(x)$ generates the sequence $(b_n)_{n=0}^{\infty}$ with $b_n = \alpha\, p_n$ for $n \geq 0$.

(c) For $\alpha \in \mathbb{R}$, the function $f(\alpha\, x)$ generates the sequence $(c_n)_{n=0}^{\infty}$ with $c_n = \alpha^n\, p_n$ for $n \geq 0$.

(d) For an integer $m \geq 0$, the function $x^m\, f(x)$ generates the sequence $0, 0, \ldots, 0, p_0, p_1, \ldots$, where the sequence starts with m 0's. I.e., $x^m\, f(x)$ generates the sequence $(d_n)_{n=0}^{\infty}$ with

$$d_n = \begin{cases} 0, & \text{if } 0 \leq n \leq m-1, \\ p_{n-m}, & \text{if } n \geq m. \end{cases}$$

(e) For any integer $m \geq 0$, the function $f(x^m)$ generates the sequence $p_0, 0, \ldots, 0, p_1, 0, \ldots, 0, p_2, 0, \ldots$ where every time we have $m-1$ 0's between the p's. I.e., $x^m\, f(x)$ generates the sequence $(e_n)_{n=0}^{\infty}$ with

$$e_n = \begin{cases} 0, & \text{if } n \geq 0 \text{ and } n \text{ is not divisible by } m, \\ p_{n/m}, & \text{if } n \geq 0 \text{ and } n \text{ is divisible by } m. \end{cases}$$

(f) The function $f'(x)$ generates the sequence $(f_n)_{n=0}^{\infty}$ with $f_n = (n+1)\, p_{n+1}$ for $n \geq 0$.

(g) The function $\displaystyle\int_0^x f(t)\, dt$ generates $(g_n)_{n=0}^{\infty}$ with $g_0 = 0$ and $g_n = \dfrac{p_{n-1}}{n}$ for $n \geq 1$.

2 Suppose $f(x)$ generates the sequence $(a_n)_{n=0}^{\infty}$. Give an expression, in terms of f, for the generating functions of the following sequences:

(a) $a_0, 0, a_1, 0, a_2, 0, \ldots$;

(b) $1, a_0, a_1, a_2, \ldots$;

(c) $a_0, -a_1, a_2, -a_3, a_4, \ldots$.

3 If the sequence $(a_n)_{n=0}^{\infty}$ is generated by the function $f(x)$, then what sequence is generated by the function $g(x) = f(0) + f(x^2) - x^2\, f(x)$?

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4 Let $f(x)$ be the generating function for the sequence $(c_n)_{n=0}^{\infty}$.

(a) Find the sequence, in terms of c_n, whose generating function is $(1-x)\,f(x)$.

(b) Find the sequence, in terms of c_n, whose generating function is $x\,f'(x)+f(x)$.

5 Find the sequences generated by the following functions:

(a) $f(x)=\dfrac{x^3}{1+x}$;

(b) $g(x)=\dfrac{x}{1-7\,x+12\,x^2}$;

(c) $h(x)=\dfrac{x^7}{2-x^7}$;

(d) $j(x)=(1+2\,x)^{15}+\dfrac{x+1}{x^2-3\,x+2}$.

6 (a) Find the generating function for the sequence $(b_n)_{n=0}^{\infty}$ given by $b_n=n^2$ for $n\geq 0$.

(b) Find the generating function for the sequence $(c_n)_{n=0}^{\infty}$ given by $c_n=n^2\,2^n$ for $n\geq 0$.

7 **Use generating functions** to find the solution of the recurrence relation

$$\begin{aligned} a_0 &= 0, \qquad a_1 = 3, \\ a_n &= 5\,a_{n-1}-6\,a_{n-2}, \qquad \text{for } n\geq 2. \end{aligned}$$

8 Let a_n be the number of n-letter words formed from the 26 letters of the alphabet, in which the five vowels A,E,I,O,U together occur an even number of times. (By a word, we mean simply a string of letters.) For example, when $n=8$, such a word is $APQIITOW$, since four (an even number) of the positions contain vowels.

(a) Show that $a_1=21$ and that $a_n=16\,a_{n-1}+5\cdot 26^{n-1}$ for $n\geq 2$.

What would you say that a_0 is?

(b) Find the generating function for the sequence $(a_n)_{n=0}^{\infty}$.

(c) Use this generating function to find a_n.

9 Again we suppose that $f(x)$ is the generating function for the sequence $(c_n)_{n=0}^{\infty}$. Define the sequence $(d_n)_{n=0}^{\infty}$ by

$$\begin{aligned} d_0 &= c_0, \\ d_n &= d_{n-1}+c_n, \qquad \text{for } n\geq 1. \end{aligned}$$

Find the generating function of $(d_n)_{n=0}^{\infty}$ in terms of $f(x)$.

10 The language of Verwegistan has words consisting of the letters **A**, **E**, **O**, **U**, **B**, **P**, and **X**. Words are formed according to the following rules: the vowels (**AEOU**) always appear in pairs of the form **AA**, **EE**, **OO**, or **UU**, and they appear in a word before all non-vowels (if any). So for instance **AAEEPXP** and **AAAA** are words, but **UUUB**, **AAXBAAX**, and **AEXX** are not.

Let v_n denote the number of words of length n.

(a) Show that $v_0 = 1$, $v_1 = 3$, and $v_n = 4\,v_{n-2} + 3^n$ for $n \geq 2$.

Let $f(x)$ be the generating function of the sequence $(v_n)_{n=0}^{\infty}$.

(b) Show that $f(x) = \dfrac{1}{(1-3\,x)\,(1-4\,x^2)}$.

(c) Use the generating function to find a closed form expression for v_n.

11 Let d_n denote the number of selections of n letters from $\{a, b, c\}$, with repetitions allowed, in which the letter a is selected an even number of times. (Note that that these selections are unordered.)

(a) Determine d_0 and d_1.

Recall from previous notes that the total number of unordered selections of n letters from $\{a, b, c\}$ with repetitions allowed is equal to $\dbinom{2+n}{n} = \dbinom{n+2}{2}$.

(b) Use this result to prove that for $n \geq 2$,

$$d_n \;=\; \binom{n+2}{2} - d_{n-1} \;=\; \tfrac{1}{2}\,(n+2)\,(n+1) - d_{n-1}.$$

(c) Show that $(d_n)_{n=0}^{\infty}$ has generating function $f(x) = \dfrac{1}{(1+x)\,(1-x)^3}$.

(d) Use the generating function to prove that $d_n = \begin{cases} \frac{1}{4}\,(n+2)^2, & \text{if } n \text{ is even,} \\ \frac{1}{4}\,(n+1)\,(n+3), & \text{if } n \text{ is odd.} \end{cases}$

12 Let $(a_n)_{n=0}^{\infty}$ be the sequence defined by the following recurrence relation:

$$a_0 \;=\; 0, \qquad a_1 \;=\; 0, \qquad \text{and}$$

$$a_n \;=\; \begin{cases} 2\,a_{n/2} + n - 1, & \text{if } n \text{ is even,} \\ a_{\lfloor n/2 \rfloor} + a_{\lceil n/2 \rceil} + n - 1, & \text{if } n \text{ is odd,} \end{cases} \qquad \text{for } n \geq 2.$$

Show that the generating function of this sequences satisfies:

$$f(x) \;=\; \big(2 + x + \frac{1}{x}\big)\,f(x^2) + \frac{x^2}{(1-x)^2}.$$

(You can use the observation that $\displaystyle\sum_{n=0,\ n \text{ even}}^{\infty} a_{n/2}\,x^n = \sum_{n=0}^{\infty} a_n\,x^{2\,n}$. And you can also use that $\displaystyle\sum_{n=2}^{\infty} (n-1)\,x^n = \frac{x^2}{(1-x)^2}$.)