CSC 2110 Discrete Mathematics
Tutorial #4 Solution

Ex1.8

7. **a)** Since $\log x$ grows more slowly than x, $x^2 \log x$ grows more slowly than x^3, so the first term dominates. Therefore this function is $O(x^3)$ but not $O(x^n)$ for any $n < 3$.

b) We know that $\log x$ grows so much more slowly than x that *every* power of $\log x$ grows more slowly than x. Thus the first term dominates, and the best estimate is $O(x^3)$.

c) By long division, we see that $f(x) = x +$ lower order terms. Therefore this function is $O(x)$, so $n = 1$.

d) Again by long division, this quotient has the form $f(x) = 1 +$ lower order terms. Therefore this function is $O(1)$. In other words, $n = 0$.

9. On the one hand we have $x^2 + 4x + 17 \leq x^2 + x^2 + x^2 = 3x^2 \leq 3x^3$ for all $x > 17$, so $x^2 + 4x + 17$ is $O(x^3)$. On the other hand, if x^3 were $O(x^2 + 4x + 17)$, then we would have $x^3 \leq C(x^2 + 4x + 17) \leq 3Cx^2$ for all sufficiently large x. But this says that $x \leq 3C$, clearly impossible for the constant C to satisfy for all large x. Therefore x^3 is not $O(x^2 + 4x + 17)$.

21. **a)** First we note that $\log(n^2 + 1)$ and $\log n$ are in the same big-O class, since $\log n^2 = 2 \log n$. Therefore the second term here dominates the first, and the simplest good answer would be just $O(n^2 \log n)$.

b) The first term is in the same big-O class as $O(n^2(\log n)^2)$, while the second is in a slightly smaller class, $O(n^2 \log n)$. (In each case, we can throw away the smaller order terms, since they are dominated by the terms we are keeping—this is the essence of doing big-O estimates.) Therefore the answer is $O(n^2(\log n)^2)$.

c) The only issue here is whether 2^n or n^2 is the faster-growing, and clearly it is the former. Therefore the best big-O estimate we can give is $O(n^{2^n})$.

33. We are given that $|f(x)| \leq C|g(x)|$ for all $x > b$ (we cannot use the variable name k here since we need it later). Hence $|f^k(x)| = |f(x)|^k \leq C^k|g^k(x)|$ for all $x > b$, so $f^k(x)$ is $O(g^k(x))$ (take the constants in the definition to be C^k and b).

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39. Let us look at the ratio

$$\frac{n!}{2^n} = \frac{n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1}{2 \cdot 2 \cdot 2 \cdots 2 \cdot 2 \cdot 2} = \frac{n}{2} \cdot \frac{n-1}{2} \cdot \frac{n-2}{2} \cdots \frac{3}{2} \cdot \frac{2}{2} \cdot \frac{1}{2}.$$

Each of the fractions in the final expression is greater than 1 except the last one, so the entire expression is at least $(n/2)/2 = n/4$. Since $n!/2^n$ increases without bound as n increases, $n!$ cannot be bounded by a constant times 2^n, which tells us that $n!$ is not $O(2^n)$.