CSC 2110 - Discrete Mathematics
Tutorial 6 - Solutions

Question 1 (Ex 3.3 5)

There are many correct answers for these questions. We will give what we consider to be the simplest ones.

a) Cleary each term in this sequence is 6 greater than the preceding term. Thus we can define the sequence by setting $a_1 = 6$ and declaring that $a_{n+1} = a_n + 6$.

b) This is just like part (a), in that each term is 2 more than its predecessor. Thus we have $a_1 = 3$ and $a_{n+1} = a_n + 2$ for all $n \geq 1$.

c) Each term is 10 times its predecessor. Thus we have $a_1 = 10$ and $a_{n+1} = 10a_n$ for all $n \geq 1$.

d) Just set $a_1 = 5$ and declare that $a_{n+1} = a_n$ for all $n \geq 1$.

Question 2 (Ex 3.3 23)

a) Since we can generate all the even integers by starting with 0 and repeatedly adding or subtracting 2, a simple recursive way to define this set is as follows: $0 \in S$; and if $x \in S$ then $x + 2 \in S$ and $x - 2 \in S$.

c) The positive integers not divisible by 5 are the ones congruent to 1, 2, 3, or 4 modulo 5. Therefore we can proceed just as in part (b), setting $1 \in S$, $2 \in S$, $3 \in S$, and $4 \in S$ as the base cases, and then declaring that if $x \in S$, then $x + 5 \in S$.

Question 3 (Example 4.1 14)

Let P by the total number of possible passwords, and let P_6, P_7, and P_8 denote the number of possible passwords of length 6, 7, and 8, respectively. By the sum rule, $P = P_6 + P_7 + P_8$. We will now find P_6, P_7, and P_8. Finding P_6 directly is difficult. To find P_6 it is easier to find the number of strings of uppercase letters and digits that are six characters long, including those with no digits, and subtract from this the number of strings with no digits. By the product rule, the number of strings of six characters is 36^6, and the number of strings with no digits is 26^6. Hence,

$P_6 = 36^6 - 26^6 = 2,176,782,336 - 308,915,776 = 1,867,866,560.$

Similarily, it can be shown that

$P_7 = 36^7 - 26^7 = 70,332,353,920$

and

$P_8 = 36^8 - 26^8 = 2,612,282,842,880.$

Consequently, $P = P_6 + P_7 + P_8 = 2,684,483,063,360.$

Question 4 (Ex 4.1 19)

This problem deals with the set of positive integers between 100 and 999, inclusive. Note that there are exactly $999 - 100 + 1 = 900$ such numbers. A second way to see this is to note that to specify a three-digit number, we need to choose the first digit to be nonzero (which can be done in 9 ways) and then the second and third digits (which can each be done in 10 ways), for a total of $9 * 10 * 10 = 900$ ways, by the product rule. A third way to see this (perhaps most relevant for this problem) is to note that a number of the desired form is a number less than or equal to 999 (and there are 999 such numbers) but not less than or equal to 99 (and there are 99 such numbers); therefore there are

$999 - 99 = 900$ numbers in the desired range.

a) Every seventh number — 7, 14, and so on — is divisible by 7. Therefore the number of positive integers less than or equal to n and divisible by 7 is $\lfloor n/7 \rfloor$ (the floor function is used — we have to round down — because the first six positive integers are not multiples of 7; for example there are only $\lfloor 20/7 \rfloor = 2$ multiples of 7 less than or equal to 20). So we find that there are $\lfloor 999/7 \rfloor = 142$ multiples of 7 not exceeding 999, of which $\lfloor 99/7 \rfloor = 14$ do not exceed 99. Therefore there are exactly 142-14=128 numbers in the desired range divisible by 7.

b) This is similar to part (a), with 7 replaced by 2, but with the added twist that we want to count the numbers not divisible by 2. Mimicking the analysis in part (a), we see that there are $\lfloor 999/2 \rfloor = 499$ even numbers not exceeding 999, and therefore 999-499=500 odd ones; there are similarly $99 - \lfloor 99/2 \rfloor = 50$ odd numbers less than or equal to 99. Therefore there are 500-50=450 odd numbers between 100 and 999 inclusive.

c) There are just 9 possible digits that a three-digit number can start with. If all of its digits are to be the same, then there is no choice after the leading digit has been specified. Therefore there are 9 such numbers.

d) This is similar to part (b), except that 2 is replaced by 4. Following the analysis there, we find that there are $999 - \lfloor 999/4 \rfloor = 750$ positive itnegers less than or equal to 999 not divisible by 4, and $99 - \lfloor 99/4 \rfloor = 75$ such positive integers less than or equal to 99. Therefore there are 750-75=675 three-digit integers not divisible by 4.

e) The method is similar to that used in the earlier parts. There are $\lfloor 999/3 \rfloor - \lfloor 99/3 \rfloor = 300$ three-digit numbers divisible by 3, and $\lfloor 999/4 \rfloor - \lfloor 99/4 \rfloor = 225$ three-digit numbers divisible by 4. Moreover there are $\lfloor 999/12 \rfloor - \lfloor 99/12 \rfloor = 75$ numbers divisible by both 3 and 4, i.e., divisible by 12. In order to count each number divisible by 3 or 4 once and only once, we need to add the number of numbers divisible by 3 to the number of numbers divisible by 4, and then subtract the number of numbers divisible by both 3 and 4 so as not to count them twice. Therefore the answer is 300+225-75=450.

f) In part (e) we found that there were 450 three-digit integers that are divisible by either 3 or 4. The others, of course, are not. Therefore there are 900-450=450 three-digit integers that are not divisible by either 3 or 4.

g) We saw in part (e) that there are 300 three-digit numbers divisible by 3 and that 75 of them are also divisible by 4. The remainder of those 300 numbers, therefore, are not divisible by 4. Thus the answer is 300-75=225.

h) We saw in part (e) that there are 75 three-digit numbers divisible by both 3 and 4.

Question 5 (Ex 4.6 13)

Assuming that the warehouses are distinguishable, let w_i be the number of books stored in warehouse i. Then we are asked for the number of solutions to the equation $w_1 + w_2 + w_3 = 3000$. By Theorem 2, there are $C(3 + 3000 - 1, 3000) = C(3002, 3000) = C(3002, 2) = 4,504,501$ of them.

Question 6 (Ex 4.6 21)

Let $d_1, d_2, \ldots, d_6$ be the digits of a number less than 1,000,000; they can each be anything from 0 to 9 (in other words, we may as well assume that there are leading 0's if necessary to make the number exactly 6 digits long). If we want the sum of the digits to equal 19, then we are asknig for the num-

ber of solutions to the equation $d_1 + d_2 + \cdots + d_6 = 19$ with $0 \leq d_i \leq 9$ for each i. Ignoring the upper bound restriction, there are, by Theorem 2, $C(6 + 19 - 1, 19) = C(24, 19) = C(24, 5) = 42505$ of them. We must then subtract the number of solutions in which the restriction is violated. If the digits are to add up to 19 and one or more of them is to exceed 9, then exactly one of them will have to exceed 9, since $10 + 10 > 19$. There are 6 ways to choose the digit that will exceed 9. Once we have made that choice (without loss of generality assume it is d_1 that is to be made greater than or equal to 10), then we coutn the number of solutions to the equation by counting the number of solutions to $d_1' + d_2 + \cdots + d_6 = 19 - 10 = 9$; by Theorem 2 there are $C(6 + 9 - 1, 9) = C(14, 9) = C(14, 5) = 2002$ of them. Thus there are $6 * 2002 = 12012$ solutions that violate the restriction. Subtracting this from the 42504 solutions altogether, we find that 42504-12012=30492 is the answer to the problem.

Question 7 (Ex 4.6 35)

We can describe any such travel in a unique way by a sequence of 4 x's, 3 y's, and 5 z's. By Theorem 3, there are

$$\frac{12!}{4!3!5!} = 27720$$

such sequences.