Selfish Routing and Path Coloring in All-Optical Networks⋆

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Abstract. We study routing and path coloring problems in all-optical networks as non-cooperative games. We especially focus on oblivious payment functions, that is, functions that charge a player according to her own strategy only.

We first strengthen a known relation between such games and online routing and path coloring. In particular, we show that the price of anarchy of such games is lower-bounded by, and in several cases precisely equal to, the competitive ratio of appropriate modifications of the First Fit algorithm.

Based on this framework we provide results for two classes of games in ring networks: in Selfish Routing and Path Coloring a player must determine both a routing and a coloring for her request, while in Selfish Path Coloring the routing is predetermined and only a coloring of requests needs to be specified. We prove specific upper and lower bounds on the price of anarchy of these games under various payment functions.

1 Introduction

In all-optical networks, communication requests are carried out by assigning to them a path in the network (routing) as well as a transmission wavelength. By using wavelength division multiplexing (WDM) it is possible to route several requests through the same link(s) of the network, and carry them out simultaneously by assigning a different wavelength to each request.

In this context, given a network topology and a set of communication requests, several interesting questions arise. If the routing of the requests is also given, the Path Coloring (PC) problem asks for the minimum number of colors (wavelengths) required such that requests sharing a common link are assigned different colors. If the routing of the requests is not given, the Routing and Path Coloring (RPC) problem asks for both a routing and a color assignment minimizing

⋆ Research supported by "Pythagoras" grant of the Greek Ministry of Education, co-funded by the European Social Fund (75%) and National Resources (25%) — Operational Program for Educational and Vocational Training II (ΕΠΕΑΕΚ II).

J. Janssen and P. Prałat (Eds.): CAAN 2007, LNCS 4852, pp. 71–84, 2007.

the number of colors under the same constraint. More optimization questions can be stated by introducing additional parameters and constraints. During the last decades a large body of work has been concentrated on the complexity and approximability questions for these optimization problems [1,2,3,4] (for a nice survey of early results see [5] and references therein).

A recent research direction concerns network optimization under game-theoretic criteria [6,7,8]. In such a context, an optimization problem can be modeled as a non-cooperative game of independent entities (players). These entities have their own objectives; they do not necessarily have to obey to a centralized protocol or they can manipulate this protocol (e.g. by providing false information) in order to achieve their own goals. The algorithmic game theory approach is used to optimize global objective functions taking into account the selfish behavior of the participating entities.

Following this direction we study the PC and RPC problems in all-optical networks as non-cooperative games. Each communication request is considered as a player and a payment function charges each player a cost depending on the (routing and color) choices of all players (including her own choices). Given a set of choices for all players we say that the game is in an equilibrium if no player can decrease her own cost by changing her choices. This equilibrium concept was first introduced by John Nash [9] and it is known as a Nash equilibrium. Although Nash has shown that each non-cooperative game has a mixed Nash equilibrium, the existence of a pure one is an open question for many games. Moreover, due to the selfish behavior of the players, such a pure equilibrium does not necessarily optimize a global objective goal. Such a goal is also known as social cost and for our problems can be defined as the number of colors used for (routing and) coloring a given set of requests. The global performance of Nash equilibria is measured by the Price of Anarchy (PoA) or coordination ratio which is defined as the ratio of the social cost of the worst Nash equilibrium over the optimal centralized solution [6], and reflects the loss in the global performance due to lack of coordination between players.

In this paper we study selfish PC and RPC in all-optical networks of ring topology; let us mention that, as far as we know, selfish PC has not been considered before. We first prove some general properties that further clarify the relation between selfish (R)PC and online (R)PC; the most important one is that the PoA of (R)PC under any *oblivious collision-free* payment function f is not smaller than the competitive ratio of a modification of the First-Fit algorithm that uses f as a selection criterion. (Note that the notion of oblivious collision-free payment function includes all functions that guarantee that no color collisions occur, but apart from that charge a player according to the player's own strategy only.) This property allows to obtain lower bounds on the PoA from lower bounds on the competitive ratio of First-Fit and its modifications; to the best of our knowledge no such lower bounds have been presented before for games in all-optical networks. We then study selfish PC and propose a payment function with PoA between 5.4 and 9. Finally, we propose two quite natural payment functions for selfish RPC. For the first of them, which forces players to choose the smallest possible color, we show a tight upper bound for the PoA which is half the trivial upper bound $\frac{|R|}{OPT}$, where R is the given set of requests and OPT is the value of an optimal centralized solution for the corresponding RPC instance. For the second, which forces players to choose shortest path routing, we give an upper bound for its PoA which does not depend on the number of players but only (logarithmically) on the size of the network. Although a payment function with PoA bounded by a constant was already known [10] our payment functions are more natural.

The paper is organized as follows: In the next section we describe the formal model for our problems and the notation used in the paper, while in Section 3 we give a brief review of related work. In Section 4 we examine the relation between the solutions obtained by online and offline algorithms for PC and RPC and the Nash equilibria for the corresponding non-cooperative games. In Sections 5 and 6 we study selfish PC and RPC, respectively; we define payment functions yielding Nash equilibria and we present upper and lower bounds for the Price of Anarchy in both cases. We conclude in Section 7 by giving a brief comparison to earlier techniques and results.

2 Game Theoretic Model

We are given a network (graph) $G = (V, E)$ and a set of communication requests R. Each request r is a pair of nodes of G, i.e., $r = (x, y)$. When the routing of requests in R is also given in advance (pre-determined) we simply consider that a set of paths P is given instead of R. Therefore, an instance of the RPC problem is denoted by (G, R) and an instance of the PC problem, where players only have to choose a color for their paths, is denoted by (G, P).

In selfish RPC (selfish PC) on G each player i issues a request r_i (a path resp.). For simplicity, we identify a player with a request. A strategy σ_i for player i is a pair (p_i, c_i) (just (c_i) for selfish PC), where p_i is a simple path connecting the endpoints of r_i and c_i is a color assigned to p_i. Let S_i denote all possible strategies of player i. The possible strategies for each player are implicated by the topology of graph G and the number of colors allowed. If we restrict the number of colors to be no more than $|R|$ then there is a finite number of strategies for each player (we do not need to define them explicitly). There is also a payment function for each player i, that is: $f_i : S_1 \times \ldots \times S_{|R|} \rightarrow \mathbb{N}$. From now on, we will restrict our study to games where all players have the same payment function f.

Definition 1. *By* S-RPC *we denote the class of Selfish-RPC games, and a game in* S-RPC *with input graph G, set of requests R, and payment function f, is denoted by a triple* (G, R, f)*.*

By S-PC *we denote the class of Selfish-PC games (pre-determined routing), and a game in* S-PC *with input graph G, set of routed requests P, and payment function f, is denoted by a triple* (G, P, f)*.*

Given a class of graphs $\mathcal{G}$ and a payment function f, we denote by S-RPC $(\mathcal{G}, f)$ *(*S-PC$(\mathcal{G}, f)$*) the subclass of* S-RPC *(*S-PC *resp.) that consists of games* (G, R, f) *((G, P, f) resp.) such that* $G \in \mathcal{G}$*.*

For a game (G, R, f) (and similarly for a game (G, P, f)) we define the following:

- A *pure strategy profile*, or simply strategy profile, is a vector $S = \{\sigma_1, \sigma_2, \ldots, \sigma_{|R|}\}$ of strategies, one for each player.
- A (pure) strategy profile is a *pure Nash Equilibrium (NE)* if for each player i it holds that

$$f(\sigma_1, \ldots, \sigma_i, \ldots, \sigma_{|R|}) \leq f(\sigma_1, \ldots, \sigma_i', \ldots, \sigma_{|R|})$$

  for any strategy $\sigma_i' \in S_i$.
- The *social cost* $sc(S)$ of strategy profile S is the number of colors used for (routing and) coloring, if no color collisions appear; otherwise $sc(S) = \infty$.

Let OPT denote the optimum social cost for a game, that is, $OPT = \min_{S \in \mathcal{S}} sc(S)$, where $\mathcal{S}$ is the set of all possible strategy profiles. Note that OPT coincides with the cost of an optimal solution of the corresponding RPC (PC) instance.

The *price of anarchy* (PoA) of a game is the worst-case number of colors used in a NE (social cost) divided by OPT, that is,

$$\text{Price of Anarchy} = \frac{\max_{S \text{ is NE}} sc(S)}{OPT}.$$

The *price of stability* (PoS) of a game is the best-case number of colors used in a NE (social cost) over OPT, that is,

$$\text{Price of Stability} = \frac{\min_{S \text{ is NE}} sc(S)}{OPT}.$$

The price of anarchy (stability) of the class of games S-RPC$(\mathcal{G}, f)$ (S-PC$(\mathcal{G}, f)$) is the maximum price of anarchy (resp. stability) among all games in S-RPC$(\mathcal{G}, f)$ (resp. S-PC$(\mathcal{G}, f)$).

Definition 2. *We say that a payment function for a selfish (routing and) path coloring game is* oblivious collision-free *if:*

(a) *it guarantees that in a Nash Equilibrium no color collisions occur (by charging a very large amount to players that use the same color and share links of the network) and*
(b) *it charges a player (who does not collide with other players) according to the player's own strategy only.*

Let us observe that for any instance of S-RPC (S-PC) with oblivious collision-free payment function it holds that $sc(S) \leq |R|$ ($sc(S) \leq |P|$, resp.) if S is a NE; hence, $PoA \leq \frac{|R|}{OPT}$ ($PoA \leq \frac{|P|}{OPT}$, resp.). All functions considered in this paper are oblivious collision-free. For the sake of simplicity we will omit from the descriptions of our payment functions the condition that guarantees collision-free Nash Equilibria.

3 Previous Work

Bilò and Moscardelli [11] consider the existence and performance of Nash equilibria of selfish RPC games in all-optical networks. They study four possible payment functions. They show that only two of these payment functions, namely when each player pays for her own color and when she pays for the maximum color used by any other overlapping player, guarantee convergence to a pure NE. However, they prove that the PoA is as high as $|R|$ even for rings. In [10] they refine this result to $\frac{|R|}{OPT}$ for any payment function which is a non-decreasing function of the color of the player.

Bilò et al. [10] consider different information levels of local knowledge that players may have for computing their payments in selfish RPC games and give bounds for the PoA in chains, rings and trees. In the complete level of information each player knows all other players' routing and coloring strategies. In the intermediate level of information each player only knows which colors are used on any edge of the network and in the minimal level of information each player knows which colors are used only on edges along paths that the player can choose. For the complete level they prove that the PoA is the same as the best approximation ratio for RPC, thus 1 in chains and 2 in rings, under payment functions specifically constructed according to the corresponding algorithms. For the intermediate level they give a payment function specifically constructed according to Slusarek's algorithm [12] for online PC in rings (also known as online circular arc coloring) that results in a PoA that is $3 + O(\frac{\log L}{L})$ in chains and $6 + O(\frac{\log L}{L})$ in rings, where L is the maximum load. For the minimal level they prove that for any payment function which is a non-decreasing function of the color of the player, the PoA in chains is bounded by the competitive ratio (say FF_{chain}) of the First-Fit algorithm for online PC in chains and the PoA in trees is $O(\log |R|)$; they also give a payment function for rings with PoA bounded by $2 \cdot FF_{chain}$. Pemmaraju et al. [13] have recently shown that $FF_{chain} \leq 8$, therefore the ratios obtained in [10] are in fact 8 in chains and 16 in rings (instead of 25.72 and 51.44 originally mentioned).

The existence of Nash equilibria and the complexity of recognizing and computing a Nash equilibrium for selfish RPC under several payment functions are considered by Georgakopoulos et al. [14]. Their results indicate that recognizing a Nash equilibrium can be done in polynomial time, when each player pays for her own color, when she pays for the maximum color used by any other overlapping player and when she pays for the most loaded edge that she uses. On the other hand, when the player pays for all the different colors appearing along her path, recognizing a Nash equilibrium is NP-complete.

4 Solutions to PC and RPC as Nash Equilibria

In this section we explore the relation of the solutions obtained by online and offline algorithms for PC and RPC to Nash equilibria for S-PC and S-RPC with respect to various oblivious collision-free payment functions.

In the online version of RPC problem requests arrive as an ordered sequence $\langle R \rangle = \langle r_1, r_2, \ldots, r_{|R|} \rangle$. Such an online instance of RPC is denoted by $(G, \langle R \rangle)$. Upon arrival of a request r_i, an online algorithm should decide a path and a color assignment to r_i so that no color collisions appear on any edge of paths that are already colored (that is, corresponding to requests r_j with $j < i$); the algorithm has no knowledge of requests that are going to appear later (that is, requests r_j with $j > i$). The objective is to minimize the number of colors used. As before, an instance of online PC is denoted by $(G, \langle P \rangle)$, where $\langle P \rangle$ is a sequence of paths ordered by arrival time.

Probably the simplest online algorithm for PC is First-Fit, which colors each request r_i with the smallest available color, provided that no color collisions occur. We will also make use of the following version of First-Fit, which is appropriate for online RPC: the algorithm chooses a path and color for request r_i in such a way that no color collisions occur and the color assigned to r_i is the minimum possible.

We now define a useful generalization of First-Fit for RPC. Consider a cost function f which specifies a cost for each path and color assignment (p, c) to a request r_i, taking into account the path and color assignment to requests $r_j,\ j < i$. Then, First-Fit with criterion f ($FF(f)$ for short) assigns to each request r_i the path p and color c that minimize $f(r_i, p, c)$, breaking ties arbitrarily. For example, the standard First-Fit for RPC described above can be seen as $FF(f)$, where $f(r_i, p, c) = c$ if p does not overlap with any path of color c, otherwise $f(r, p, c) = \infty$. A similar generalization of First-Fit for PC is defined analogously by using cost functions that take into account only the color assignment to requests (since paths are given; in this case the payment function has two arguments p and c). Formally, in the above description for each payment function f, the path-color (or just color) assignment to requests $r_j,\ j < i$, should also appear as argument of function f; we will omit it here for the sake of simplicity.

The following two lemmata reveal an interesting relation between selfish routing and coloring and the corresponding online (centralized) problems. The second lemma is in fact a slight reformulation of an observation from [10].

Lemma 1. *Consider a game (G, R, f) in* S-RPC *(*S-PC*) where f is an oblivious collision-free payment function. For any ordering $\langle R \rangle$ of R, an execution of $FF(f)$ algorithm on $(G, \langle R \rangle)$ gives a strategy profile for (G, R, f) which is a Nash Equilibrium.*

Proof. Consider the path-color assignment obtained by an execution of $FF(f)$ on $(G, \langle R \rangle)$. A request r_i cannot be assigned a path-color combination of lower cost unilaterally, otherwise $FF(f)$ would have chosen that path-color combination for r_i. The reason is that if such a different assignment is possible then it does not cause color collisions with respect to the path-color assignment of all other requests. Therefore, it certainly does not cause any color collision with respect to requests $r_j, j < i$; hence, upon arrival of r_i, $FF(f)$ would have chosen this lower cost assignment. □

Lemma 2 ([10]). *Consider a game (G, R, f) in* S-RPC *(*S-PC*) where f is collision-free and non-decreasing on the players' color (hence also oblivious). For every strategy profile S that is a Nash Equilibrium for (G, R, f), there is an ordering $\langle R\rangle$ of R such that there is an execution of $FF(f)$ algorithm on $(G, \langle R\rangle)$ yielding the same path-color assignment to R as S.*

We now show how to convert any (routing and) coloring solution to RPC (PC) to a Nash Equilibrium for the corresponding game in S-RPC (S-PC resp.) with at most the same number of colors.

Lemma 3. *Let k be the number of colors used in a solution to instance (G, R) of* RPC *((G, P) of* PC *respectively). We can compute a strategy profile which is a Nash Equilibrium of social cost at most k for game (G, R, f) in* S-RPC *(*S-PC *respectively) where f is oblivious collision-free and a non decreasing function of the players' color.*

Proof. We convert the solution to instance (G, R) for RPC into a strategy profile which is a Nash Equilibrium for game (G, R, f) in S-RPC by using the Nash Conversion algorithm described below.

Algorithm 1. *Nash Conversion*

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for each color c := 1 to k do
  for each request r colored with c do
    for each color c' := 1 to c − 1 do
      if there exists a path (including the current one) for request r that does not
         overlap with any other path colored with c'
      then { use that path to route r and color it with c'; exit for }
```

For PC the above algorithm works by modifying the "if" statement as follows: "if the path of r does not overlap with any path colored with c' then assign color c' to r".

Note that no request can move to a smaller color, because Algorithm 1 assigns the smallest available color, say c', to r and does not affect afterwards the path-color assignment of requests that have color smaller than c'. □

Combining the above lemmata we obtain the following theorem:

Theorem 1. *Let $\mathcal{G}$ be a class of graphs.*

1. *The price of anarchy for the class of games* S-RPC$(\mathcal{G}, f)$ *(*S-PC$(\mathcal{G}, f)$*), where f is oblivious collision-free, is at least as large as the competitive ratio of $FF(f)$ for* RPC *(*PC*, resp.) in graphs that belong to $\mathcal{G}$.*
2. *The price of anarchy for the class of games* S-RPC$(\mathcal{G}, f)$ *(*S-PC$(\mathcal{G}, f)$*), where f is oblivious collision-free and is a non-decreasing function of the players' color, is equal to the competitive ratio of First-Fit for* RPC *(*PC*, resp.) in graphs that belong to $\mathcal{G}$.*

3. *The price of stability for any game* (G, R, f) *in* S-RPC *(*S-PC*), where* f *is oblivious collision-free and is a non-decreasing function of the players' color, is equal to* 1.

Proof. 1: By Lemma 1, each execution of $FF(f)$ leads to a path-color assignment which is a NE for a game in S-RPC$(\mathcal{G}, f)$ (S-PC$(\mathcal{G}, f)$); the social cost of that NE is equal to the number of colors used by $FF(f)$. Dividing by OPT we get the claim.

2: Let S be a NE of the highest social cost. By Lemma 2, it turns out that there is an execution of $FF(f)$ on the corresponding RPC (PC) instance that requires the same number of colors as S. Dividing by OPT we get that the competitive ratio of $FF(f)$ is at least as large as the price of anarchy for S-RPC$(\mathcal{G}, f)$ (S-PC$(\mathcal{G}, f)$ resp.). Combining with 1 we get the claim.

3: It suffices to consider the optimal coloring and convert it to a NE by using the Nash Conversion algorithm. □

Using Theorem 1.2 and the fact that the competitive ratio of the First Fit algorithm for online PC in chains is between 4.4 and 8 [13], we have that:

Corollary 1. *The payment function* $f(p, c) = c$ *induces* S-PC *games in chains with a price of anarchy between* 4.4 *and* 8.

5 S-PC in Rings

In this section we propose and study an oblivious collision-free payment function that results in a relatively low PoA for S-PC in rings.

We first observe that the natural choice of taking as payment function the one that charges the color value gives $2.53 \log n + 5$ (n is the number of nodes) on the PoA for S-PC in rings. This is obtained by Theorem 1.2 and the competitive ratio of the First Fit algorithm for online PC in rings shown in [15].

Let L_e be the load on edge e, i.e. number of paths that use e. Let E' be a set of edges then we denote by $L_{E'}$ the maximum load over all edges in E'. Let L be the maximum load of G. Consider an arbitrary edge e of ring G. Payment function f_e is defined as follows: if a player p (recall that players can be seen as paths in this case) uses edge e then she is encouraged to use the smallest available color, since she pays the value of the color she uses; otherwise she is encouraged to use the smallest available color which is greater than L_e (for which she pays the color value) instead of using any color in $\{1, \ldots, L_e\}$ (for which she must pay a much higher price). Formally,

$$f_e(p, c) = x_{p,e} \times \lfloor \frac{L_e}{c} \rfloor \times |P| + c$$

where $x_{p,e} = \begin{cases} 0 \text{ , if } p \text{ traverses edge } e \\ 1 \text{ , otherwise} \end{cases}$

Recall that FF_{chain} denotes the competitive ratio of First-Fit for online PC in chains (online interval coloring).

Theorem 2. *The payment function f_e induces* S-PC *games in rings with a price of anarchy equal to $FF_{chain} + 1$.*

Proof. Observe that the graph $G-e$ is a chain. Payment function f_e implies that all players that have their path in path set $P_{G-e} = P \setminus P_e$ have no gain by using a color from the set $\{1, \ldots, L_e\}$, because it results in high cost (at least $|P|$). Therefore, a Nash equilibrium S can be seen as the result of two independent executions of First-Fit on players: (a) the first execution is on players (paths) that use e, with available colors $\{1, \ldots, L_e\}$, and (b) the second execution is on players that do not use e, with available colors $\{L_e + 1, \ldots, |P|\}$. Both subsets of players are ordered according to their color in S (increasingly).

For the first group of players L_e colors will be used, while for the second, of load L_{G-e}, First Fit will need at most $FF_{chain} \cdot L_{G-e}$ colors. Hence, the total number of colors in S will be

$$\begin{aligned} sc(S) &\leq L_e + FF_{chain} \cdot L_{G-e} \leq (FF_{chain} + 1) \max\{L_e, L_{G-e}\} \\ &\leq (FF_{chain} + 1) OPT \end{aligned}$$

because any algorithm will need at least as many colors as the maximum load of requests. Since no specific assumption was made for S, the above inequality holds for all NE implying that $PoA \leq FF_{chain} + 1$.

It is also possible to bound PoA from below by considering an instance where P_e consists of L requests and P_{G-e} consists of a worst-performance chain instance for First-Fit of load L. Assume that paths in P_e do not overlap paths in P_{G-e}. Then $OPT = L$. On the other hand, if we give this instance as input to $FF(f_e)$ algorithm, with the requests in P_{G-e} ordered as in the worst-performance instance of First-Fit, then $FF(f_e)$ will need $(FF_{chain} + 1)L$ colors. By Theorem 1.1 this implies that $PoA \geq FF_{chain} + 1$. □

Corollary 2. *The payment function f_e induces* S-PC *games in rings with a price of anarchy between* 5.4 *and* 9*.*

6 S-RPC in Rings

In this section we consider S-RPC in rings induced by two different oblivious collision-free payment functions. The first one forces players to choose the smallest possible color while the second one forces them to choose shortest path routing.

6.1 The Color-Length Payment Function

We consider the payment function $f(r, p, c) = c \cdot n + length(p)$, where n is the number of nodes in the ring. It is clear that under this function a player r always selects the smallest possible color even if it requires to follow the longest one of her two possible alternative paths.

Theorem 3. *The payment function function $f(r, p, c) = c \cdot n + length(p)$ induces* S-RPC *games in rings with a price of anarchy equal to $\frac{|R|}{2\ OPT} + 1$, where R is the given set of requests.*

Proof. We first prove that $PoA \leq \frac{|R|}{2\ OPT} + 1$.

Let S be a NE for a $(Ring, R, f)$ game. Let R_1 be the subset of requests assigned an exclusive color (assigned only to one of these requests) and R_2 be the subset of requests that share a color with at least one other request. It follows that

$$sc(S) \leq \frac{|R_2|}{2} + |R_1| = \frac{|R|}{2} + \frac{|R_1|}{2}.$$

We shall prove that $\frac{|R_1|}{2} \leq OPT$ and therefore

$$PoA = \frac{sc(S)}{OPT} \leq \frac{|R|}{2\ OPT} + 1.$$

Clearly the requests in R_1 are routed via paths that overlap each other, for otherwise at least two of them can take the same color. Moreover, each of them is routed via its shortest path for otherwise S would not be a NE for $(Ring, R, f)$. This is because such a request, routed via its longest path, can improve its own cost by choosing its shortest path and keeping its (unique) color. Hence, the requests in R_1 are routed via paths of length at most $n/2$.

Consider the optimal solution; it uses OPT colors. According to this coloring the set of requests, R, can be partitioned into OPT disjoint subsets $C_1, C_2, \ldots, C_{OPT}$, each one containing the requests assigned the same color. The requests in each $C_i,\ 1 \leq i \leq OPT$, are routed via non overlapping paths and hence they are consecutive in a clockwise traversal of the ring i.e., no request starts or ends between the start and the end point of any other. Therefore, they, but at most one, are routed via their shortest paths; that is, at most one of them is routed via a path of length greater than $n/2$.

Consider now the routing of the requests in R_1 in S and in the optimal solution. In both cases the routing of these requests coincides (shortest path routing) except for at most OPT requests i.e., the single requests that are possibly routed via longest paths in each set $C_i,\ 1 \leq i \leq OPT$. Since the requests in R_1 are routed via paths overlapping each other, it follows that at most two requests from each set C_i can be in R_1: the one that is routed via its longest path, now routed via its shortest path, and one of the rest. Therefore, $|R_1| \leq 2\ OPT$.

We prove next, by a counterexample, that $PoA \geq \frac{|R|}{2\ OPT} + 1$.

Consider the following instance: A ring of $2k+6t$ nodes and a set R of requests consisting of $k+2$ subsets $R_0, \ldots, R_{k+1}$, each containing t requests:

- R_0 consists of t 'crossing' requests $\{k+2t+(j-1), (2k+5t+j)\}, 1 \leq j \leq t$ (the last request is between node $k+3t-1$ and node $2k+6t$).
- $R_i, 1 \leq i \leq k$, consists of t identical requests $\{i, i+1\}$ (from node i to node $i+1$).
- R_{k+1} consists of t 'crossing' requests $\{k+j, k+j+t\}, 1 \leq j \leq t$.

The optimal solution routes R_1 to R_{k+1} with shortest paths and R_0 with longest paths, and assigns to each R_i colors $\{1, \ldots, t\}$ (see Figure 1). Thus $OPT = t$.

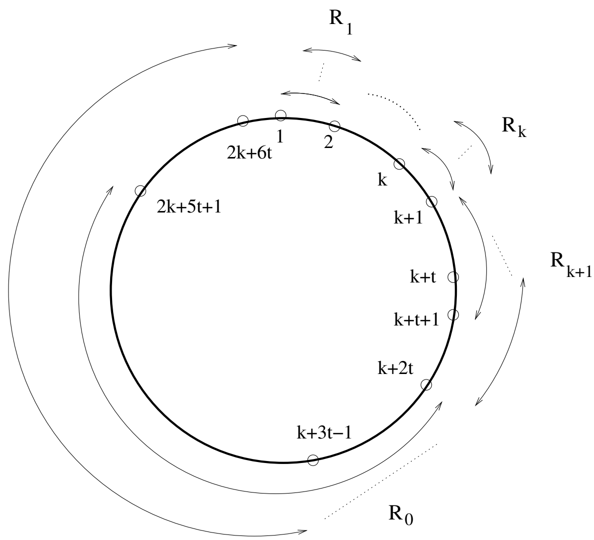

Fig. 1. Example of a S-RPC game and its optimal centralized solution

Consider now the execution of $FF(f)$ algorithm on instance $(Ring, \langle R \rangle)$ of online RPC, assuming that requests in R_i appear in $\langle R \rangle$ before requests in R_j for $i < j$. Then, $FF(f)$ first routes requests in R_0 via their shortest paths, that is, via $\{1, k+2t\}$, and assigns them colors $\{1, \ldots, t\}$. Then, for each $R_i, 1 \leq i \leq k$, every two requests are routed via complementary paths and receive the same color; thus $t/2$ new colors are needed for each $R_i, 1 \leq i \leq k$ (see Figure 2). Finally, requests in R_{k+1} would overlap each other and every other previously considered request, no matter which of the two possible paths is used. Therefore t new colors are needed for requests R_{k+1} and the shortest path is chosen for each of them. Altogether, $FF(f)$ uses $\frac{kt}{2} + 2t$ colors. By Theorem 1.1 the number of colors used by $FF(f)$ is a lower bound of the social cost of the worst Nash equilibrium, that is

$$PoA \geq \frac{\frac{kt}{2} + 2t}{t} = \frac{(k+2)t}{2t} + 1 = \frac{|R|}{2OPT} + 1.$$

□

6.2 The Length-Color Payment Function

We consider the payment function $f(r, p, c) = length(p) \cdot |R| + c$, where R is the given set of requests. It is clear that under this function a player r always selects the shortest one of its two possible alternative paths even if it requires to take a larger color.

Theorem 4. *The payment function $f(r, p, c) = length(p) \cdot |R| + c$ induces* S-RPC *games in rings with a price of anarchy such that $FF_{chain} + 1 \leq PoA \leq 5.06 \log n + 10$, where n is the number of nodes in the ring.*

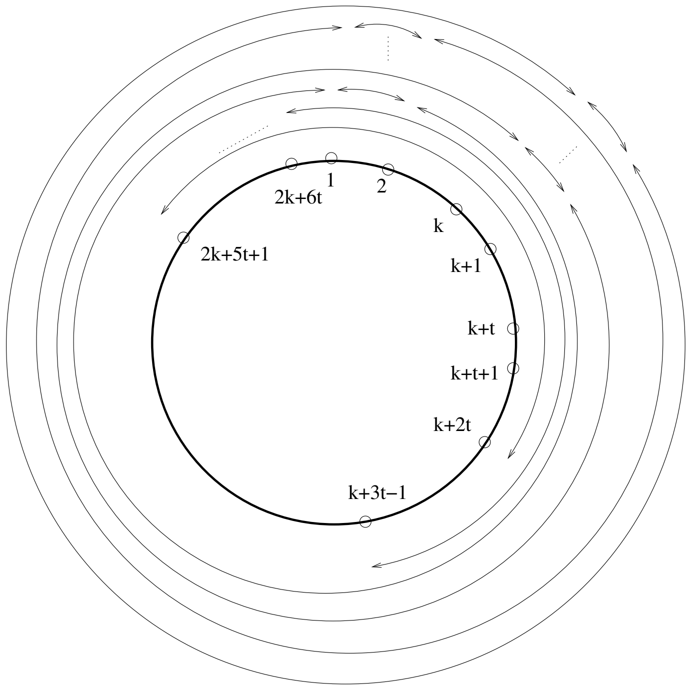

Fig. 2. Solution obtained by applying algorithm FF(color-length) on the instance of Figure 1 (only subsets R_i, $0 \leq i \leq k$, are shown, for the sake of clarity)

Proof. The proof of the upper bound is based on a result for dynamic wavelength assignment on rings in [15]. This result states that the First Fit algorithm needs at most $2.53L \log n + 5L$ wavelengths, where L is the maximum load on the ring. Combining this result with the observation that any shortest path routing produces a maximum load that is at most twice the one produced by an optimal routing we get the upper bound.

For the proof of the lower bound we consider the following S-RPC game: A ring of $2k$ nodes and a set R of requests consisting of 2 subsets R_1 and R_2 such that:

- R_1 consists of an arbitrary number of requests which, when routed via their shortest paths, yield a maximum load of L in the ring links. Moreover, for each request $(i, j) \in R_1$ it holds that $1 \leq i \neq j \leq a < k - 1$.
- R_2 consists of L identical requests $(1, i)$, $k > i > a$.

The optimal solution routes R_1 via shortest and R_2 via longest paths. This way no request in R_1 overlaps with any request in R_2. Requests in R_1 require L colors, since they are on the chain $1, 2, \ldots, a$, and requests in R_2 can be colored by the same L colors. Therefore, $OPT = L$.

The $FF(f)$ online algorithm on instance $(Ring, \langle R \rangle)$ of RPC, if requests in R_1 appear in $\langle R \rangle$ before requests in R_2 (or vice versa), routes all requests via shortest paths and therefore it uses $L \cdot FF_{chain}$ colors for R_1 and L new colors for R_2. Using Theorem 1.1 it follows that $PoA \geq FF_{chain} + 1$. □

7 Conclusions

In this paper we studied selfish (routing and) path coloring games in all-optical networks. We proposed a payment function for S-PC in rings with a PoA between 5.4 and 9. For S-RPC in rings we studied two natural payment functions: one called 'color-length' which favors smallest colors and one called 'length-color' which favors shortest paths. We have shown that the color-length function fails to achieve a low PoA; however, its PoA is half the PoA of any payment function that charges according to the value of the color only [10,11]. On the other hand, the length-color function achieves a PoA which does not depend on the number of requests but only on the number of nodes of the ring (logarithmically). It is still open whether the upper bound for the length-color function can be further improved taking into account that the lower bound we have shown is as low as 5.4. Note that all our functions require only local color information, namely to know which colors are used along edges that can be used by a player (minimal level of information using the classification in [10]).

Comparing to earlier work we observe that, as far as we know, S-PC has not been considered before. For S-RPC in rings, a payment function with $PoA \leq 16$ has been proposed in [10]; however, that payment function forces players to avoid routing through a particular edge of the graph, which may increase the total traffic of the network. Therefore, our length-color function might be more appropriate in cases where reducing the total traffic is important (e.g. if the social cost takes into account the sum of the loads over all edges).

In order to obtain our results we established a connection of the PoA of selfish (routing and) path coloring games to the competitive ratio of First-Fit-like algorithms for the corresponding online (routing and) path coloring problems. This connection is a generalized and strengthened form of an observation from [10]. In particular, the observation in [10] was used in order to obtain an upper bound on PoA from the competitive ratio of First-Fit. Our strengthening allows to obtain lower bounds as well.

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